Analytic Option Prices for the Black-Karasinski Short Rate Model*

Blanka Horvath†, Antoine Jacquier†, Colin Turfus‡

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Abstract

We consider a one-parameter family of short rate models which encompasses both Hull-White (normal) and Black-Karasinski (lognormal) models. We deduce a general form for the relevant Green's function as an asymptotic series, assuming only that the deviations of the short rate from the forward curve are on average small in absolute terms, and show how this solution can be parametrised in such a way as to fit the model to a term structure of zero coupon bond prices. We use the derived Green's function to calculate conditional bond prices and pricing formulae for caps and floors to second order accuracy. The results are seen to take a form which is straightforward to compute using quadrature and even the first order expressions achieve highly favourable comparison with benchmark Monte Carlo computations for a wide range of market conditions with both long and short cap/floor maturities.

1 Introduction

Over a quarter of a century has elapsed since Black and Karasinski (1991) first proposed their eponymous lognormal short rate model for interest rates as an alternative to the normal model of Hull and White (1990) which, although relatively tractable, does not guarantee positive rates. As observed by Brigo and Mercurio (2006), the Black-Karasinski model turned out to be rather less tractable, which "renders the model calibration to market data more burdensome than in the Hull and White (1990) Gaussian model, since no analytical formulas for bonds are available." More recently, Turfus (2016a) demonstrated how the distribution of conditional prices of zero coupon bonds under the Black-Karasinski model can be calculated consistent with a condition of no arbitrage, with errors which are third order in the short rate, with none of the assumptions about the smallness of volatility which are customarily made. Turfus (2016b) has shown how analytic expressions for caplet prices can also be derived for a family of models which includes both Hull-White and Black-Karasinski under an assumption of low volatility levels. This latter calculation was based on an asymptotic modelling approach to stochastic rates first propounded by Kim and Kunitomo (1999) in the context of vanilla equity option pricing. Their approach allowed them to address affine short rate models, in particular those of Hull and White (1990) and of Cox, Ingersoll and Ross (1985).

In the present paper, we build on the above-mentioned results. We adopt the approach first taken by Hagan et al. (2005) in expounding their celebrated SABR model, namely to derive an asymptotically valid Green's function for the underlying pricing PDE and to use this to deduce power series representations of option prices. Surprisingly little direct use has been made of this approach since then, although recently it has been revived and used by Pagliarani et al. (2011) in the context of local-stochastic volatility modelling. They have together with a number of collaborators applied their method *inter alia* to pricing Asian options on equity or other lognormal underlyings under a local volatility assumption (Foschi et al., 2013). We follow

*The views expressed herein should not be considered as investment advice or promotion. They represent personal research of the authors and do not purport to reflect the views of their employers (current or past), or the associates or affiliates thereof.

†Department of Mathematics, Imperial College, London. b.horvath@imperial.ac.uk
†Department of Mathematics, Imperial College, London. a.jacquier@imperial.ac.uk
‡Deutsche Bank, London. colin.turfus@db.com

a similar approach in a short rate model context in deriving a second-order accurate asymptotic Green's function for the Black-Karasinski and related models, then showing how this can be applied to bond and option pricing.

We introduce in §2 a one-parameter family of beta blend models which includes Black-Karasinski and also the Hull-White model as special cases. In §3 we go on to derive a representation of the requisite Green's function which is asymptotically valid for small deviations of the short rate from the forward curve, illustrating in the process how to calibrate the model to a term structure of zero coupon bond prices. The second-order accurate Green's function obtained is then applied in §4 to the problem of calculating conditional bond prices to second order accuracy and further to obtain pricing formulae for caps/floors and capped/floored Libor payments. Favourable comparisons are found with Monte Carlo simulations in §5. Conclusions are stated in §6 and mention made of a number of ways the asymptotic approach proposed can be extended into a multi-factor modelling context to price hybrid derivatives.

2 Modelling Assumptions

We consider a beta blend short rate process depending on a parameter for the purpose of pricing interest rate derivatives. The parameter β may be taken to be constant or more generally time-dependent. This model was first proposed in the literature by Turfus (2016c) to describe the credit default intensity process in a hybrid equity-credit model and subsequently used also by Turfus (2016b) in an interest rate context. It encompasses both Hull and White (1990) and Black and Karasinski (1991) models as special cases, respectively for $\beta \uparrow 1$ and $\beta = 0$.

In quantifying the short rate, we shall find it convenient to work with an ancillary process x_t satisfying the following canonical Ornstein-Uhlenbeck processes:

$$dx_t = -\alpha_r(t)x_t dt + \sigma_r(t)\, dW_t \tag{1}$$

with dW_t a Wiener process. Here the mean reversion parameter $\alpha_r(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and the volatility $\sigma_r(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ are taken to be piecewise continuous functions. We follow the convention here and below that processes are distinguished from functions of t by use of a subscript t for the former. This ancillary process is related to the interest short rate r_t by

$$(1-\beta(t))\, r_t + \beta(t)\bar{r}(t) = (\bar{r}(t) + (1-\beta(t))r^*(t))\mathcal{E}(x_t, t) \tag{2}$$

where we have defined $\mathcal{E}(x,t) = \mathcal{E}(x,t,t)$ with[1]

$$\mathcal{E}(x,t,u) := \exp\left(F_\beta(u)\phi_r(t,u)x - \tfrac{1}{2}F_\beta^2(u)\phi_r^2(t,u)I_r(0,t)\right), \quad t \geq 0 \tag{3}$$

$$F_\beta(t) := \frac{(1-\beta(t))}{(\bar{r}(t) + \delta(1-\beta(t)))^{\beta(t)}}, \quad t \geq 0 \tag{4}$$

$$\phi_r(t,u) := e^{-\int_t^u \alpha_r(s)ds}, \quad u \geq t, \tag{5}$$

$$I_r(s,t) := \int_s^t \phi_r^2(u,t)\sigma_r^2(u)du, \quad t \geq s \geq 0. \tag{6}$$

Here $\bar{r}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is assumed to be continuous, $\beta(\cdot) : \mathbb{R}^+ \to [0,1)$ is a parameter[2] controlling the skew of the model and $\delta \geq 0$ is a small parameter which can be introduced if needed to ensure positivity of the denominator. We further assume that $x_0 = 0$, with $t = 0$ the 'as of' date for which the model is calibrated. The function $r^*(t)$ is determined by calibration but will tend in the zero volatility limit to zero. The formal no-arbitrage constraint which determines this function is as follows

$$E\left[e^{-\int_0^t r_s\, ds}\right] = D(0,t) \tag{7}$$

[1] It may be observed that $\mathcal{E}(x_t,t,u)$ thus defined is the stochastic exponential of $F_\beta(t)\phi_r(t,u)x_t$, $t \geq 0$.
[2] We shall see below how results for $\beta \equiv 1$ can be conveniently recovered by taking the limit as $\beta \to 1^-$.

under the equivalent martingale measure for $0 < t \leq T_m$, where T_m is the longest maturity date for which the model is calibrated, and

$$D(t_1, t_2) := e^{-\int_{t_1}^{t_2} \overline{r}(s)\, ds} \tag{8}$$

is the t_1-forward price of the t_2-maturity zero coupon bond.

We consider the (stochastic) time-t price of a European-style security which pays a cash amount $P(x_T)$ at maturity T, denoting this by $f_t = f(x_t, t)$. We note in particular that the price of a T-maturity zero coupon bond is obtained by taking $P(x) = 1$.[3] We infer by standard means that in the general case $f(x,t)$ satisfies the following backward diffusion equation:[4]

$$\frac{\partial f}{\partial t} - \alpha_r(t)x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_r^2(t)\frac{\partial^2 f}{\partial x^2} = r(x,t)f. \tag{9}$$

subject to the final condition $\lim\limits_{t\to T^-} f(x,t) = P(x)$, with $r(x,t) := r_t|_{x_t=x}$ and re-express this in operator notation as

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - h(x,t)\right) f(x,t) = 0 \tag{10}$$

where

$$\mathcal{L}[\cdot\,] := \frac{1}{2}\sigma_r^2(t)\frac{\partial^2}{\partial x^2} - \alpha_r(t)x\frac{\partial f}{\partial x} - \overline{r}(t), \tag{11}$$

and $h(x,t) := h(x,t,t)$ with

$$h(x,t,u) := \frac{(\overline{r}(u) + (1-\beta(u))r^*(u))\mathcal{E}(x,t,u) - \overline{r}(u)}{1-\beta(u)}, \quad u \geq t, \tag{12}$$

We further propose that the impact of $h(.)$, considered as an operator, on $f(.)$ is $\mathcal{O}(\epsilon)$ in comparison to that of $\mathcal{L}$ and consider the limit as $\epsilon \to 0$. Effectively we assume that the deviations of r_t from the forward curve (or equivalently those of Libor rates) are small in absolute terms.

Following Turfus (2016a), we note that satisfaction of the calibration condition (7) imposes conditions on $r^*(\cdot)$ at $\mathcal{O}(\epsilon^2)$ and higher. On that basis we propose an asymptotic expansion

$$r^*(t) = \sum_{j=2}^{\infty} r_j^*(t), \tag{13}$$

with $r_j^*(t) = \mathcal{O}(\epsilon^j)$. Substituting for $r^*(\cdot)$ into (12), we infer a similar expansion for $h(x,t)$:

$$h(x,t) = \sum_{j=1}^{\infty} h_j(x,t), \tag{14}$$

with $h_j(x,t) = \mathcal{O}(\epsilon^j)$. In particular

$$h_1(x,t) := \frac{\overline{r}(t)(\mathcal{E}(x,t)-1)}{1-\beta(t)}. \tag{15}$$

$$h_j(x,t) := r_j^*(t)\mathcal{E}(x,t), \quad j \geq 2. \tag{16}$$

In the following section, we will look to derive a Green's function solution for (10) as an asymptotic power series in ϵ.

[3] We will look to derive the functional form of $f(\cdot)$ implied by our model in this case and in the process to determine the conditions on $r^*(t)$ necessary to satisfy (7).

[4] Here the coefficient of f is simply the short rate r_t conditional on $x_t = x$.

3 Green's Function

3.1 Derivation of Green's Function

We will be interested in the so-called free-boundary Green's function solution to (10) which tends to zero as $x \to \pm\infty$. We consider first the limiting problem with $\epsilon \equiv 0$. The required limit solution can be written

$$G_0(x,t;\xi,v) = D(t,v)\frac{\partial}{\partial \xi} N\left(\frac{\xi - \phi_r(t,v)x}{\sqrt{I_r(t,v)}}\right), \tag{17}$$

where $N(\cdot)$ is a unit Gaussian cumulative probability distribution function. Guided by Kato (1995, see in particular Chapter IX, §2), we deduce the following result:

Theorem 3.1 *The function $G(x,t;\xi,v)$ defined by the asymptotic series*

$$G(x,t;\xi,v) = \sum_{n=0}^{\infty} G_n(x,t;\xi,v) \tag{18}$$

with $G_0(x,t;\xi,v)$ given by (17) and $G_n(x,t;\xi,v) = \mathcal{O}(\epsilon^n)$ defined recursively by

$$G_n(x,t;\xi,v) := -\sum_{i=1}^{n} \int_t^v \int_{-\infty}^{\infty} G_0(x,t;x_1,t_1) h_i(x_1,t_1) G_{n-i}(x_1,t_1;\xi,v)\, dx_1 dt_1, \quad n \geq 1 \tag{19}$$

is a Green's function solution for (10) satisfying

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - h(x,t)\right) G(x,t;\xi,v) = \delta(x-\xi)\delta(t-v) \tag{20}$$

subject to $G(x,t;\xi,v) \to 0$ as $x \to \pm\infty$ for $0 \leq t \leq v$.

Proof 3.1 *For the proof of Theorem 3.1, see Appendix A*

In particular for $n = 1$ we obtain from (19):

$$\begin{aligned} G_1(x,t;\xi,v) &= -\int_t^v \int_{-\infty}^{\infty} G_0(x,t;x_1,t_1) h_1(x_1,t_1) G_0(x_1,t_1;\xi,v)\, dx_1 dt_1 \\ &= -\int_t^v \frac{\overline{r}(t_1)}{1-\beta(t_1)} \left(\mathcal{E}(x,t,t_1)\mathcal{M}_{t_1} - 1\right) G_0\left(x,t;\xi,v\right) dt_1, \end{aligned} \tag{21}$$

where we have defined the shift operator

$$\mathcal{M}_u G_0(x,t;\xi,v) := G_0\left(x,t;\xi - F_\beta(u)\phi_r(u,v)I_r(t,u),v\right), \tag{22}$$

and for $n = 2$:

$$
\begin{aligned}
G_2(x,t;\xi,v) &= -\int_t^v \int_{-\infty}^{\infty} G_0(x,t;x_2,t_2)h_1(x_2,t_2)G_1(x_2,t_2;\xi,v)\,dx_2dt_2 \\
&\quad -\int_t^v \int_{-\infty}^{\infty} G_0(x,t;x_1,t_1)h_2(x_1,t_1)G_0(x_1,t_1;\xi,v)\,dx_1dt_1 \\
&= \int_t^v \frac{\overline{r}(t_2)}{1-\beta(t_2)} \left(\mathcal{E}(x,t,t_2)\mathcal{M}_{t_2}-1\right) \int_{t_2}^v \frac{\overline{r}(t_1)}{1-\beta(t_1)} \left(\mathcal{E}(x,t,t_1)\mathcal{M}_{t_1}-1\right) G_0(x,t;\xi,v)\,dt_1dt_2 \\
&\quad + \int_t^v \frac{\overline{r}(t_2)}{1-\beta(t_2)} \mathcal{E}(x,t,t_2) \int_{t_2}^v \frac{\overline{r}(t_1)}{1-\beta(t_1)} \mathcal{E}(x,t,t_1) \left(e^{F_\beta(t_2)F_\beta(t_1)\phi_r(t_2,t_1)I_r(t,t_2)}-1\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{M}_{t_1}\mathcal{M}_{t_2}G_0(x,t;\xi,v)\,dt_1dt_2 \\
&\quad - \int_t^v r_2^*(t_1)\mathcal{E}(x,t,t_1)\mathcal{M}_{t_1}G_0\left(x,t;\xi,v\right)dt_1 \\
&= \int_t^v \frac{\overline{r}(t_2)}{1-\beta(t_2)} \int_t^{t_2} \frac{\overline{r}(t_1)}{1-\beta(t_1)} \left(\mathcal{E}(x,t,t_1)\mathcal{M}_{t_1}-1\right)\left(\mathcal{E}(x,t,t_2)\mathcal{M}_{t_2}-1\right) G_0(x,t;\xi,v)\,dt_1dt_2 \\
&\quad + \int_t^v \frac{\overline{r}(t_2)}{1-\beta(t_2)} \mathcal{E}(x,t,t_2) \int_t^{t_2} \frac{\overline{r}(t_1)}{1-\beta(t_1)} \mathcal{E}(x,t,t_1) \left(e^{F_\beta(t_1)F_\beta(t_2)\phi_r(t_1,t_2)I_r(t,t_1)}-1\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{M}_{t_1}\mathcal{M}_{t_2}G_0(x,t;\xi,v)\,dt_1dt_2 \\
&\quad - \int_t^v r_2^*(t_1)\mathcal{E}(x,t,t_1)\mathcal{M}_{t_1}G_0\left(x,t;\xi,v\right)dt_1,
\end{aligned} \tag{23}
$$

where in the last expression we have changed the order of integration and reversed the roles of t_1 and t_2.

3.2 Model Calibration

Completion of the formal derivation of our asymptotically valid Green's function requires that we ensure it is consistent with the no-arbitrage constraint (7), by choosing a suitable specification for the function $r_2^*(t)$. To this end we let f_t^T be the time-t value of a unit cash flow paid at time T, whence the required final condition is $f_T^T = 1$. Applying our second order accurate Green's function for $t = 0$, we obtain

$$
f_0^T = D(0,T)\left(1+\int_0^T \left(\frac{\overline{r}(t_2)}{1-\beta(t_2)} \int_0^{t_2} \frac{\overline{r}(t_1)}{1-\beta(t_1)} \left(e^{F_\beta(t_1)F_\beta(t_2)\phi_r(t_1,t_2)I_r(0,t_1)}-1\right)dt_1 - r_2^*(t_2)\right)dt_2\right)
$$

We see that satisfaction of the required no-arbitrage condition $f_0^T = D(0,T)$ at $\mathcal{O}(\epsilon)$ is automatic on the basis of no $\mathcal{O}(\epsilon)$ term in $r^*(t)$ correctly having been assumed, whereas at $\mathcal{O}(\epsilon^2)$ we must choose

$$
r_2^*(t) = \frac{\overline{r}(t)}{1-\beta(t)} \int_0^t \frac{\overline{r}(u)}{1-\beta(u)} \left(e^{F_\beta(u)F_\beta(t)\phi_r(u,t)I_r(0,u)}-1\right)du \tag{24}
$$

We see that *defining* our model by the second order accurate Green's function and choosing $r^*(t) = r_2^*(t)$ will give us exact calibration. Note that in the limit as $\beta \uparrow 1$ we obtain in this way

$$
r^*(t) \to \int_0^t \phi_r(u,t)I_r(0,u)du \tag{25}
$$

as the well-known Hull-White limit.

Finally, making use of (60) with (17), (21), (23) and (24), and supposing we can in general write

$$
P(x) = \sum_{j=0}^{\infty} P_j(x), \tag{26}
$$

with $P_j(x) = \mathcal{O}(\epsilon^j)$, we can compute a second order accurate representation of $f(x,t)$ as

$$f(x,t) \sim \sum_{i=0}^{2} \sum_{j=0}^{2-i} \int_{-\infty}^{\infty} G_i(x,t;\xi,T) P_j(\xi) d\xi. \tag{27}$$

We note in passing that, under the assumed continuity of $\overline{r}(\cdot)$, the form of our Green's function imposes no smoothness restrictions on the target payoff function $P(\cdot)$, other than that it be integrable over any finite subset of $\mathbb{R}$.

4 Applications

4.1 Zero Coupon Bond Pricing

We consider next the conditional price f_t^T of a T-maturity zero coupon bond using (27). In this case the payoff is given by $P(x) = 1$, whence we obtain

$$f_t^T = D(t,T)\left(1 - F_1^T(x_t,t) + F_2^T(x_t,t) + \mathcal{O}(\epsilon^3)\right). \tag{28}$$

where the $F_i^T(\cdot)$ are $\mathcal{O}(\epsilon^i)$ and given by[5]

$$F_1^T(x,t) = \int_t^T \frac{\overline{r}(t_1)}{1-\beta(t_1)} \left(\mathcal{E}(x,t,t_1) - 1\right) dt_1 \tag{29}$$

$$F_2^T(x,t) = \tfrac{1}{2} F_1^T(x,t)^2 + \int_t^T \mathcal{E}(x,t,t_2) \left(\frac{\overline{r}(t_2)}{1-\beta(t_2)} \int_t^{t_2} \frac{\overline{r}(t_1)}{1-\beta(t_1)} \mathcal{E}(x,t,t_1) \left(e^{F_\beta(t_1)F_\beta(t_2)\phi_r(t_1,t_2)I_r(t,t_1)} - 1\right) dt_1 - r_2^*(t_2)\right) dt_2. \tag{30}$$

In the Hull-White limit ($\beta \uparrow 1$) this becomes

$$f_t^T \sim D(t,T)\left(1 - B^*(t,T)(x + r_2^*(t)) + \tfrac{1}{2}(x^2 - I_r(0,t)) B^{*2}(t,T)\right) \tag{31}$$

where

$$B^*(t,v) := \int_t^v \phi_r(t,u) du, \tag{32}$$

which is consistent to second order with the exact result

$$f_t^T = D(t,T) \exp\left(-B^*(t,T)(x + r_2^*(t)) - I_r(0,t)) B^{*2}(t,T)\right). \tag{33}$$

4.2 Caplet Pricing

Consider a caplet which pays at time T a coupon based on the positive difference between tenor-τ Libor and a strike K on a unit notional, based on a payment period $[T-\tau, T]$, which rate we shall denote $L(\tau, T)$. For simplicity we assume no spread between forward Libor rates and the equivalent risk-free rates inferred from (2) above. However, it is not difficult to introduce an assumed deterministic spread by adjusting the value of the strike accordingly in the formulae derived below. We can express the caplet payoff at time T by

$$\text{Payoff}_T := \max\left\{(L(\tau,T) - K)\delta(T-\tau,T), 0\right\}$$

where $\delta(t_1,t_2)$ is the day count faction calculated according to the relevant convention (typically actual/360 or actual/365). We note in particular that, under our assumptions, the realised Libor rate is related to the stochastic zero coupon bond price f_t^T calculated above by

$$1 + L(\tau,T)\delta(T-\tau,T) = (f_{T-\tau}^T)^{-1} \tag{34}$$

[5] We have here used the identity that, by symmetry, $\int_t^T f(v) \int_t^v f(u)\, du\, dv = \frac{1}{2}\left(\int_t^T f(u)\, du\right)^2$.

whence

$$\text{Payoff}_T = \max\left\{(f_{T-\tau}^T)^{-1} - (1 + K\delta(T-\tau,T)), 0\right\}$$

If we consider an equivalent-valued payment made at time $T-\tau$, this must be discounted by precisely the T-maturity zero coupon bond price observed at time $T-\tau$, whence we can write

$$\text{Payoff}_{T-\tau} = \kappa^{-1}\max\left\{\kappa - f_{T-\tau}^T, 0\right\} \tag{35}$$

with

$$\kappa := \frac{1}{1 + K\delta(T-\tau,T))}. \tag{36}$$

In other words we should consider a put option on the bond price. Writing the time-t price of this option as $C^{T,K}(x_t,t)$, we see this will satisfy (10) with final condition that

$$C^{T,K}(x, T-\tau) = \text{Payoff}_{T-\tau}|_{x_{T-\tau}=x}. \tag{37}$$

We shall seek here to calculate the caplet price up to second order. The equivalent-valued payoff at time $T-\tau$ is, using (28):

$$P(x) = \max\left\{1 - \kappa^{-1}D(T-\tau,T)\left(1 - F_1^T(x,T-\tau) + F_2^T(x,T-\tau)\right), 0\right\} + \mathcal{O}(\epsilon^3). \tag{38}$$

We see that, since $F_1^T(\cdot) = \mathcal{O}(\epsilon)$, for the option to have non-trivial value in the limit as $\epsilon \to 0$, we must have that $1 - \kappa^{-1}D(T-\tau,T) = \mathcal{O}(\epsilon)$, so we make this assumption. Let us denote $\xi^* := \inf\{x \,|\, P(x) > 0\}$, in terms of which the $P_j(x)$ in (26) can be written

$$P_0(x) = 0 \tag{39}$$

$$P_1(x) = (1 - \kappa^{-1}D(T-\tau,T)(1 - F_1^T(x,T-\tau)))\mathbb{1}_{x>\xi^*} \tag{40}$$

$$P_2(x) = -\kappa^{-1}D(T-\tau,T)F_2^T(x,T-\tau)\mathbb{1}_{x>\xi^*} \tag{41}$$

Posing

$$C^{T,K}(x,t) = \sum_{j=1}^{2} C_j^{T,K}(x,t) + \mathcal{O}(\epsilon^3) \tag{42}$$

in obvious notation and making use of (27), we obtain:

$$\begin{aligned} C_1^{T,K}(x,t) &= \int_{-\infty}^{\infty} G_0(x,t;\xi,T-\tau)P_1(\xi)\,d\xi \\ &= (D(t,T-\tau) - \kappa^{-1}D(t,T))N(-d_1(\xi^*-x,t,T-\tau)) \\ &\quad + \kappa^{-1}D(t,T)\int_{T-\tau}^{T} \frac{\overline{r}(u)}{1-\beta(u)} \\ &\qquad (\mathcal{E}(x,t,u)N(-d_2(\xi^*-x,t,u,T-\tau)) - N(-d_1(\xi^*-x,t,T-\tau)))\,du \end{aligned} \tag{43}$$

where we define, for $u, w \leq v$,

$$d_1(\xi,t,w) := \frac{\xi}{\sqrt{I_r(t,w)}}, \tag{44}$$

$$d_2(\xi,t,u,w) := d_1(\xi - F_\beta(u)\phi_r(u\wedge w, u\vee w)I_r(t,u\wedge w), t, w), \tag{45}$$

$$d_2^*(\xi,t,u,v,w) := d_1(\xi - F_\beta(v)\phi_r(w,v)I_r(u\wedge w, w), t, w), \tag{46}$$

$$d_3(\xi,t,u,v,w) := d_2(\xi - F_\beta(v)\phi_r(w,v)I_r(u\wedge w, w), t, u, w), \tag{47}$$

with the binary operators $\wedge$ and $\vee$ denoting min and max, respectively. Likewise at second order, we have

$$
\begin{aligned}
C_2^{T,K}(x,t) &= \int_{-\infty}^{\infty} \left(G_1(x,t;\xi,T-\tau)P_1(\xi) + G_0(x,t;\xi,T-\tau)P_2(\xi)\right) d\xi \\
&= -(1-\kappa^{-1}D(T-\tau,T)) \int_t^{T-\tau} \frac{\overline{r}(u)}{1-\beta(u)} \left(\mathcal{E}(x,t,u)\mathcal{M}_u - 1\right) \int_{\xi^*}^{\infty} G_0(x,t;\xi,T-\tau)\, d\xi du \\
&\quad - \kappa^{-1}D(T-\tau,T) \int_t^{T-\tau} \frac{\overline{r}(u)}{1-\beta(u)} \left(\mathcal{E}(x,t,u)\mathcal{M}_u - 1\right) \int_{\xi^*}^{\infty} G_0(x,t;\xi,T-\tau) \\
&\qquad\qquad F_1^T(\xi,T-\tau)\, d\xi du \\
&\quad - \kappa^{-1}D(t,T) \int_{\xi^*}^{\infty} G_0(x,t;\xi,T-\tau) F_2^T(\xi,T-\tau)\, d\xi \\
&= -(D(t,T-\tau) - \kappa^{-1}D(t,T)) \int_t^{T-\tau} \frac{\overline{r}(u)}{1-\beta(u)} \\
&\qquad \left(\mathcal{E}(x,t,u)N(-d_2(\xi^*-x,t,u,T-\tau)) - N(-d_1(\xi^*-x,t,T-\tau))\right) du \\
&\quad - \kappa^{-1}D(t,T) \int_{T-\tau}^{T} \frac{\overline{r}(v)}{1-\beta(v)} \mathcal{E}(x,t,v) \int_t^v \frac{\overline{r}(u)}{1-\beta(u)} \\
&\qquad \Big(e^{F_\beta(u)F_\beta(v)\phi_r(u\wedge(T-\tau),v)I_r(t,u\wedge(T-\tau))}\mathcal{E}(x,t,u)N(-d_3(\xi^*-x,t,u,v,T-\tau)) \\
&\qquad\qquad - N(-d_2^*(\xi^*-x,t,u,v,T-\tau))\Big)\, dudv \\
&\quad + \kappa^{-1}D(t,T) \int_{T-\tau}^{T} \frac{\overline{r}(v)}{1-\beta(v)} \mathcal{E}(x,t,v) \int_t^v \frac{\overline{r}(u)}{1-\beta(u)} \\
&\qquad \left(\mathcal{E}(x,t,u)N(-d_2(\xi^*-x,t,u,T-\tau)) - N(-d_1(\xi^*-x,t,T-\tau))\right) dudv \\
&\quad + \kappa^{-1}D(t,T) \int_{T-\tau}^{T} r_2^*(v)\mathcal{E}(x,t,v)N(-d_2(\xi^*-x,t,v,T-\tau))\, dv. \qquad (48)
\end{aligned}
$$

Setting $x = t = 0$ and combining terms, we obtain:

$$
\begin{aligned}
C^{T,K}(0,0) &\sim \left(D(0,T-\tau) - \kappa^{-1}D(0,T)\right) N(-d_1(\xi^*,0,T-\tau)) \\
&\quad + \kappa^{-1}D(0,T) \int_{T-\tau}^{T} \frac{\overline{r}(u)}{1-\beta(u)} \left(N(-d_2(\xi^*,0,u,T-\tau)) - N(-d_1(\xi^*,0,T-\tau))\right) du \\
&\quad - (D(0,T-\tau) - \kappa^{-1}D(0,T)) \int_0^{T-\tau} \frac{\overline{r}(u)}{1-\beta(u)} \\
&\qquad \left(N(-d_2(\xi^*,0,u,T-\tau)) - N(-d_1(\xi^*,0,T-\tau))\right) du \\
&\quad - \kappa^{-1}D(0,T) \int_{T-\tau}^{T} \frac{\overline{r}(v)}{1-\beta(v)} \int_0^v \frac{\overline{r}(u)}{1-\beta(u)} e^{F_\beta(u)F_\beta(v)\phi_r(u\wedge(T-\tau),v)I_r(0,u\wedge(T-\tau))} \\
&\qquad \left(N(-d_3(\xi^*,0,u,v,T-\tau)) - N(-d_2(\xi^*,0,v,T-\tau))\right) dudv \\
&\quad + \kappa^{-1}D(0,T) \int_{T-\tau}^{T} \frac{\overline{r}(v)}{1-\beta(v)} \int_0^v \frac{\overline{r}(u)}{1-\beta(u)} (N(-d_2^*(\xi^*,0,u,v,T-\tau)) \\
&\qquad - N(-d_2(\xi^*,0,v,T-\tau)) + N(-d_2(\xi^*,0,u,T-\tau)) - N(-d_1(\xi^*,0,T-\tau)))\, dudv, \qquad (49)
\end{aligned}
$$

with $\mathcal{O}(\epsilon^3)$ errors, where the first two lines constitute the first order solution and the remainder provides the necessary second order adjustment. A resemblance to the standard Black formula is here evident in the first order solution, the most notable difference being that the $d_2(\cdot)$ argument is sampled across the integration range $u \in [T-\tau, T]$, rather than being considered fixed, as with the corresponding (exact) Hull-White result.

We see that the integrand in (49), although it is not defined for $\beta = 1$, is yet well-behaved in the limit as $\beta \uparrow 1$, which it is a straightforward matter to compute. However, given the existence of an exact solution,

this is of less interest than when $\beta < 1$. The Black-Karasinski result is obtained by setting $\beta \equiv 0$ and $F_\beta \equiv 1$ in (49); also in (29) and (30), which are needed to obtain ξ^* from (38).

4.3 Floorlet Pricing

By extension, the corresponding second-order accurate expression for the floorlet is

$$
\begin{aligned}
F^{T,K}(0,0) \sim & \left(\kappa^{-1}D(0,T) - D(0,T-\tau)\right) N(d_1(\xi^*,0,T-\tau)) \\
& - \kappa^{-1}D(0,T)\int_{T-\tau}^{T} \frac{\overline{r}(u)}{1-\beta(u)} \left(N(d_2(\xi^*,0,u,T-\tau)) - N(d_1(\xi^*,0,T-\tau))\right) du \\
& - \left(\kappa^{-1}D(0,T) - D(0,T-\tau)\right)\int_{0}^{T-\tau} \frac{\overline{r}(u)}{1-\beta(u)} \\
& \qquad \left(N(d_2(\xi^*,0,u,T-\tau)) - N(d_1(\xi^*,0,T-\tau))\right) du \\
& + \kappa^{-1}D(0,T)\int_{T-\tau}^{T} \frac{\overline{r}(v)}{1-\beta(v)} \int_0^v \frac{\overline{r}(u)}{1-\beta(u)} e^{F_\beta(u)F_\beta(v)\phi_r(u\wedge(T-\tau),v)I_r(0,u\wedge(T-\tau))} \\
& \qquad \left(N(d_3(\xi^*,0,u,v,T-\tau)) - N(d_2(\xi^*,0,v,T-\tau))\right) dudv \\
& - \kappa^{-1}D(0,T)\int_{T-\tau}^{T} \frac{\overline{r}(v)}{1-\beta(v)} \int_0^v \frac{\overline{r}(u)}{1-\beta(u)} \left(N(d_2^*(\xi^*,0,u,v,T-\tau))\right. \\
& \qquad \left. - N(d_2(\xi^*,0,v,T-\tau)) + N(d_2(\xi^*,0,u,T-\tau)) - N(d_1(\xi^*,0,T-\tau))\right) dudv, \quad (50)
\end{aligned}
$$

with the definition of ξ^* unchanged (although technically the payoff function now has the opposite sign, whence ξ^* will be a supremum for x rather than an infimum). It is also readily verified that use of (49) and (50) leads to

$$
C^{T,K}(0,0) - F^{T,K}(0,0) = D(0,T)\left(\int_{T-\tau}^{T} \overline{r}(u)\, du - K\delta(T-\tau,T)\right), \quad (51)
$$

so put-call parity is seen to be exactly satisfied under our second order approximation.

4.4 Capped/Floored Libor Flows

We consider also briefly the pricing of Libor flows which are capped and/or floored. We note that the impact of a cap at level K_C on a Libor flow is by replication given by subtracting the value of a caplet with strike K_C. Since the time-t value of a Libor flow struck at $T-\tau > t$ for payment at T is, using (28),

$$
L^T(x_t,t) = f_t^{T-\tau} - f_t^T, \quad (52)
$$

the capped value will be

$$
CL^{T,K_C}(x_t,t) = L^T(x_t,t) - C^{T,K_C}(x_t,t). \quad (53)
$$

and the corresponding value of a Libor flow floored at K_F will be

$$
FL^{T,K_F}(x_t,t) = L^T(x_t,t) + C^{T,K_F}(x_t,t). \quad (54)
$$

If both a cap and a floor are applied, the result will be

$$
CFL^{T,K_C,K_F}(x_t,t) = CL^{T,K_C}(x_t,t) + C^{T,K_F}(x_t,t). \quad (55)
$$

5 Comparison of Results

In the following we focus on the pricing of caps: given the demonstration of put-call parity in (51), the error associated with the asymptotic approximation in the corresponding floors is guaranteed to be identical, so does not merit separate investigation.

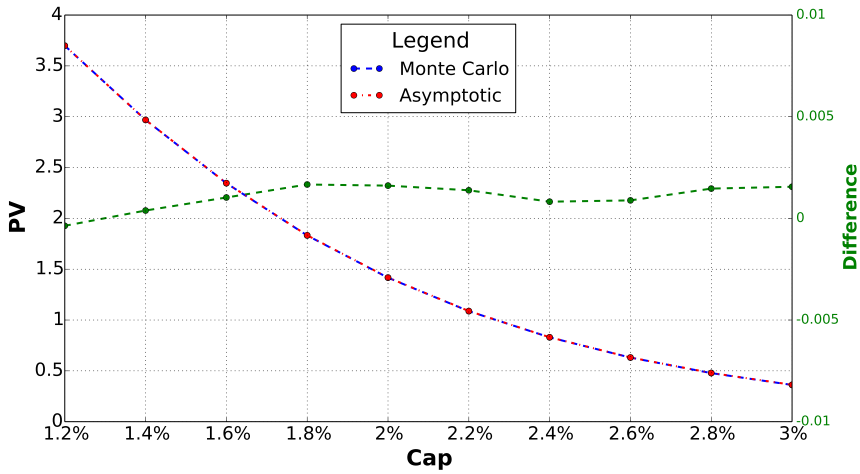

Figure 1: Black-Karasinski prices for $5y$ maturity cap with 6m Libor tenor

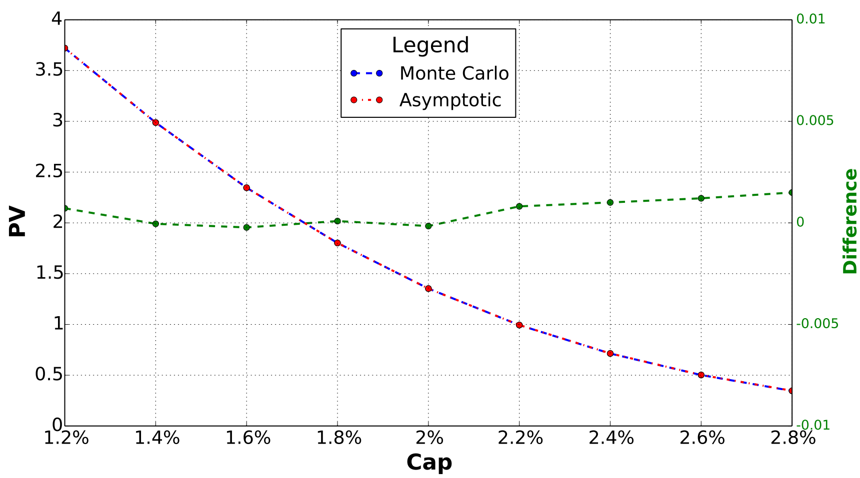

Figure 2: Blend prices with $\beta = 0.5$ for $5y$ maturity cap with 6m Libor tenor

Calculations were mainly done for the important Black-Karasinski case, using a first order asymptotic representation. That is, we ignored the contribution from F_2^T in (38) and used only the first two lines of (49). Comparisons were made with a Monte Carlo simulation using a conservative 500,000 simulations, ensuring any Monte Carlo error did not impact significantly on the quality of the comparison. The short rate is taken to follow the profile of the USD market for January 2017 rising from around 1% to around 3% over a five year period. The (lognormal) local volatility is taken to be 30%, with a (constant) mean reversion rate of 25%. The results are illustrated in Fig. 1 for various moneyness levels. As can be seen, they are indistinguishable to visual accuracy. It was observed that in-the-money caplets are slightly underpriced and the out-of-the-money caplets correspondingly overpriced, but the latter effect is considerably more marked, resulting in almost all caps being to some degree overpriced.

By way of comparison, we also consider a case with $\beta = 0.5$. On this occasion we chose a local volatility of 4%, which yields comparable prices. Results are illustrated in Fig. 2. The level of error is seen in general to be less than that for the Black-Karasinski case, with underpricing again found for in-the-money caplets and overpricing for out-of-the money caplets, the latter effect being the stronger.

Since the Black-Karasinski model is the case of greatest interest, we focus in the remaining tests on

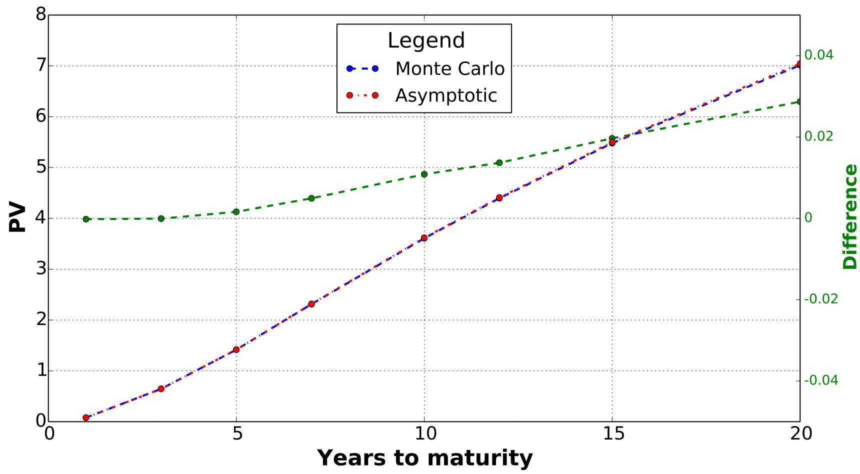

Figure 3: Black-Karasinski prices for caps with different maturities

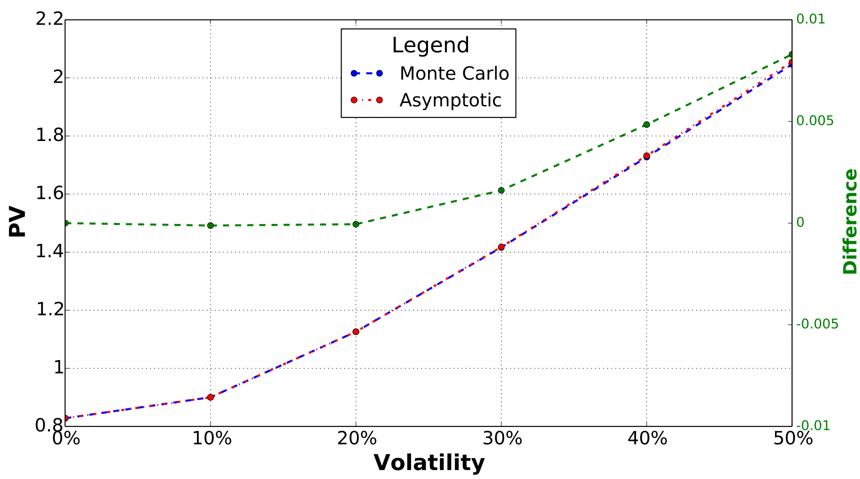

Figure 4: Dependence of Black-Karasinski prices for 5y caps on volatility level

that case. We consider next how the asymptotic expansion performs in the case of different maturities up to 20y. For the purposes of allowing comparative analysis across different maturities, we focus on ATM options by moving the strike of the caps appropriately. Results are shown in Fig. 3. They remain fairly indistinguishable to visual accuracy. As is evident, the error increases with maturity but of course so does the cap value. Overall the relative error increases, so the approximation does deteriorate slightly for longer maturities.

Next we consider the variation of cap prices with the volatility $\sigma_r(t)$. Results are shown in Fig. 4. As with time to maturity, the error is seen to grow as volatility increases, as does the price, both approximately linearly in the high volatility limit. Even at the very high level of 50% volatility the error is still only about 0.4% in relative terms.

A similar comparison is illustrated in Fig. 5 of the dependency of 5y cap prices on the assumed volatility mean reversion rate $\alpha_r(t)$. Not surprisingly, as the mean reversion rate approaches zero allowing the volatility to grow without bound as T increases, the errors in the asymptotic approximation become larger, but even in such circumstances no greater in relative terms than those observed in Fig. 4 above.

We consider the impact of the Libor tenor on 5y cap prices in Fig. 6. As is to be expected, the tenor does

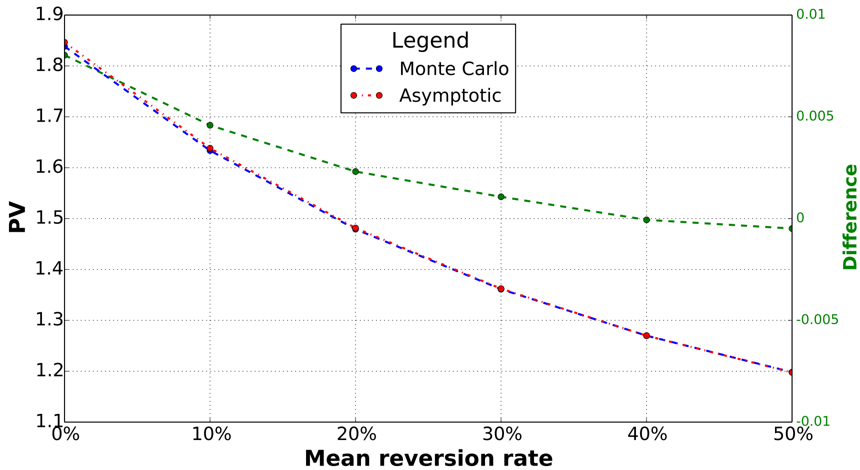

Figure 5: Dependence of Black-Karasinski prices for 5y caps on volatility mean reversion rate α_r

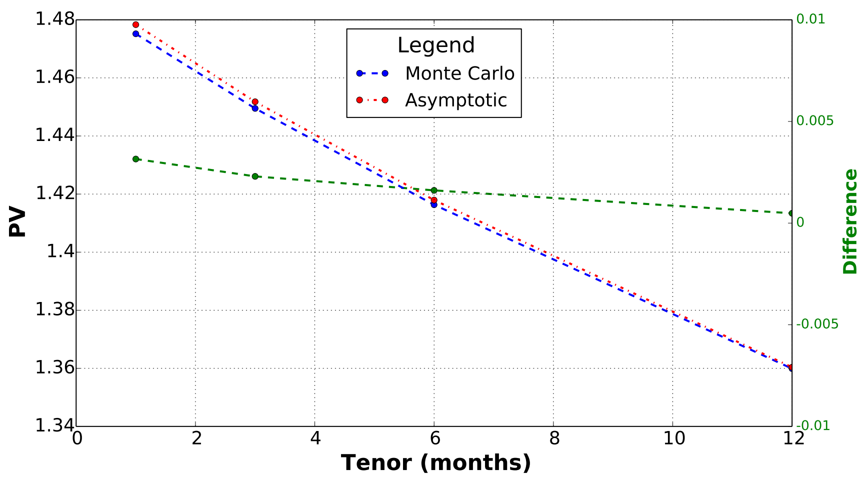

Figure 6: Dependence of Black-Karasinski prices for 5y caps on Libor tenor

not impact greatly on the price. But it does impact on the error, with the caps based on 1y Libor resulting in significantly smaller discrepancies. With the magnified y-axis in this case, the slight overpricing of the asymptotic model is visually detectable.

Given that our asymptotic expansion is based on an assumption of small fluctuations in the rates in absolute terms, it is to be expected that the accuracy will be lessened as the level of the yield curve is raised with fixed lognormal volatility levels. The impact of such a change is considered in Fig. 7, where the yield curve is parallel-bumped upward by up to 8%. To enable a fair comparison, the strike is raised by the same amount in each case, ensuring that options remained roughly at the money. As can be seen, the option value increases due to the (fixed) lognormal volatility having greater PV impact for higher rates. As expected, the error also increases, roughly in proportion. Even in the fairly extreme case of a 10% interest rate, the relative error in the 5y cap PV was seen to be only 0.5% in relative terms. In circumstances where rates exceeded the 10% level, it might be considered worthwhile to include second order terms in the calculation so as to ensure greater accuracy.

In conclusion, we infer that even our rather simple first order approximations ought to be usable for many practical purposes, particularly calibration and risk management where exact reproduction of market

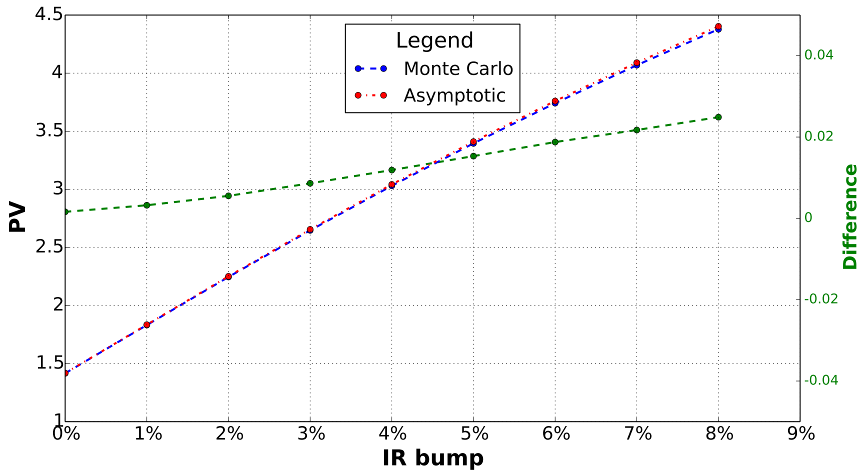

Figure 7: Dependence of ATM Black-Karasinski prices for 5y caps on IR curve level

prices is less essential than in pricing for trading purposes. If greater accuracy is required, particularly in the Black-Karasinski case, the inclusion of second order terms ought easily to provide enough accuracy with very little additional computational burden.

6 Conclusions

A new family of short rate models is introduced which offers a blend between the well-known Hull-White and Black-Karasinski models. An approximate Green's function was derived in §3 for the resultant pricing PDE, based on the assumption of the deviations of the Libor rates from their forward values being small in absolute terms. This Green's function was used to derive the second-order accurate expression (28) for the conditional price of zero coupon bonds. Further, second-order accurate expressions were derived for caplet and floorlet prices, which results can be combined to allow analytic pricing of caps and floors, or indeed of capped/floored Libor flows. The caplet price is given by (49) and the corresponding floorlet price by (50). These expressions are all straightforward to compute. Similar results can be found for the Hull-White case but are of less interest on account of availability of an exact analytic solution. Even the first order expressions were found to yield highly favourable comparison with benchmark Monte Carlo computations for a wide range of market conditions with both long and short cap/floor maturities.

The expressions for $\beta < 1$ should prove useful in calibrating the model to at-the-money market prices for caps and floors, which task would otherwise be computationally intensive. However an additional important use of the approach presented here is in the context of the short rate model being combined with another component model (credit intensity, equity, FX and/or another interest rate, say) for hybrid derivative pricing purposes. Of particular interest is use of the Black-Karasinski version of the model to describe credit intensities since this guarantees their positivity. Such work is currently under way and numerous preliminary results have been obtained. See for example Turfus (2017a) and Turfus (2018b) for application to the valuation of credit default swaps taking account of wrong-way correlation risk, Turfus (2017b) for the pricing of capped/floored Libor payments in a quanto CDS pricing context, Turfus (2017c) for the pricing of CDS on an underlying interest rate swap, Turfus (2017d) for an application to calculating the CVA on a collateralised portfolio of equity options, Turfus (2017e) for the pricing of convertible bonds and Turfus (2018a) for an example involving the pricing of quanto CDS and CDS with a quanto loss cap.

Much work has also been done by other authors using perturbation approaches to obtain analytic approximations for prices of various option types under local and/or stochastic volatility modelling assumptions. See, for example, Pagliarani et al. (2011), who considered equity option pricing under a

local volatility assumption, obtaining a perturbation expansion for the relevant Green's function much as we did here, and using it to derive asymptotic expressions for option prices. Their approach was applied also to Asian option pricing in Foschi et al. (2013) and extended, with the use of some Fourier analysis, to incorporate Lévy jumps in the dynamics of the spot underlying in Pagliarani and Pascucci (2013). A review of a number of other papers which have presented asymptotic option pricing formulae in recent years has been given by Turfus and Schubert (2017). An interesting prospect for future work would be to extend the above by applying the asymptotic short rate model for stochastic rates/credit described herein alongside a local-stochastic volatility model to capture their combined impact, as was done by Funahashi (2015) for European equity options.

A Proof of Theorem 3.1

We demonstrate that (18) is a solution to (20). We start by defining

$$G^N(x,t;\xi,v) = \sum_{n=0}^{N} G_n(x,t;\xi,v). \tag{56}$$

Using (19) and the fact that by definition

$$\left(\frac{\partial}{\partial t} + \mathcal{L}\right) G_0(x,t;\xi,v) = \delta(x-\xi)\delta(t-v) \tag{57}$$

we see that

$$\frac{\partial}{\partial t} G_n(x,t;\xi,v) = \sum_{i=1}^{n} h_i(x,t) G_{n-i}(x,t;\xi,v) - \mathcal{L} G_n(x,t;\xi,v), \quad n \geq 1, \tag{58}$$

whence, using (56) and (57),

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathcal{L}\right) G^N(x,t;\xi,v) &= \delta(x-\xi)\delta(t-v) + \sum_{n=1}^{N}\sum_{i=1}^{n} h_i(x,t) G_{n-i}(x,t;\xi,v) \\ &= \sum_{i=1}^{N} h_i(x,t) \sum_{r=0}^{N-i} G_r(x,t;\xi,v) + \delta(x-\xi)\delta(t-v). \end{aligned} \tag{59}$$

Finally, taking $G(x,t;\xi,v)$ to be $\lim_{N\to\infty} G^N(x,t;\xi,v)$ and using (56), (59) and (14) we obtain

$$\left(\frac{\partial}{\partial t} + \mathcal{L}\right) G(x,t;\xi,v) = h(x,t) G(x,t;\xi,v) + \delta(x-\xi)\delta(t-v), \tag{60}$$

confirming that $G(\cdot)$ so defined is the required Green's function. This concludes the proof of the theorem.

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